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QUADRATURES OF THE SOLUTIONS OF THE FIRST AND SECOND INITIAL-BOUNDARY-VALUE PROBLEMS OF ELASTICITY THEORY FOR AN ANISOTROPIC MATERIAL†

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A method described in a previous paper [1] is used to construct quadratures of the solutions of the first and second initial-boundary-value problems of elasticity theory for a homogeneous anisotropic body of finite dimensions with a piecewise-smooth boundary of arbitrary shape. © 2002 Elsevier Science Ltd. All rights reserved.

1. FOURIER TRANSFORMATION

We will define the Fourier transformation of the functions used as follows [2]

$$f^*(\mathbf{k}) = \int_{R^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} d\mathbf{x}$$

For a function to be expressed in terms of its Fourier transformation by the inversion formula [2]

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \left[\dots \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f^*(\mathbf{k}) e^{ix_1k_1} dk_1 \right] \dots \right] e^{ix_nk_n} dk_n \tag{1.1}$$

it must satisfy the conditions [2]

$$\begin{gathered} |f(x_1, x_2, \dots, x_k + t_k, \dots, x_n) - f(x_1, x_2, \dots, x_n)| \leqslant c_k(x_1, x_2, \dots, x_{k-1}) |t_k|^\alpha \\ c_1 = \text{const}, \quad 0 \leqslant \alpha \leqslant 1, \quad k = 1, \dots, n \\ \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} c_k(x_1, x_2, \dots, x_{k-1}) dx_1 \dots dx_{k-1} < \infty, \quad \forall k \end{gathered} \tag{1.2}$$

Theorem 1. A differentiable function possessing a Fourier transform is represented in terms of its Fourier transform by formulae (1.1).

Proof. A function having at point M a partial derivative with respect to the variable X_k satisfies the Hölder condition at the point [3], i.e. a finite constant A_K exists such that, for any α satisfying the condition $0 < \alpha \leqslant 1$, the following inequality is satisfied

$$|f(x_1, x_2, \dots, x_k + t_k, \dots, x_n) - f(x_1, x_2, \dots, x_n)| \leqslant A_k |t_k|^\alpha$$

Suppose the region of definition of G of the functions considered is bounded. Having selected the functions $c_k(x_1, \dots, x_j)$ from the condition $c_k(x_1, \dots, x_j) = A_k$, we obtain

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} c_k(x_1, x_2, \dots, x_{k-1}) dx_1 \dots dx_{k-1} = A_k \int_{G_{k-1}} dx_1 \dots dx_{k-1} < \infty, \quad \forall k$$

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i.e. the function $f(\mathbf{x})$ at point M satisfies conditions (1.2). Hence it follows that finite continuously differentiable functions are reproduced by their Fourier transform at all points of their region of definition. However, functions that are continuously differentiable in the finite region G and equal to zero outside this region are reproduced by their Fourier transform in all cases apart from the boundary of the region, where the function and its derivatives are discontinuous.

2. THE CONVOLUTION OF FUNCTIONS OVER A FINITE REGION AND ITS FOURIER TRANSFORMATION

The convolution of the functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ over finite region G will be understood as function $f(\mathbf{x})$ defined in the region $G_x \subset R^n$ by the relation

$$f(\mathbf{x}) = \int_{G_y} f_1(\mathbf{x}-\mathbf{y}) f_2(\mathbf{y}) d\mathbf{y} \tag{2.1}$$

The functions $f_1(\mathbf{x}-\mathbf{y})$ and $f_2(\mathbf{y})$ are defined in the regions G_{xy} and G_y. The region $G_y \subset R^n$ is finite, and G_{xy} is defined by the regions G_x and G_y and by the expression $(\mathbf{x}-\mathbf{y})$.

Theorem 2. Let the functions $f(\mathbf{x})$, $f(\mathbf{x}-\mathbf{y})$ and $f_2(\mathbf{y})$ be absolutely integrable in their regions of definitions and be connected by relation (2.1).

Then, for their Fourier transforms, the following relation holds

$$f^*(\mathbf{k}) = f_1^*(\mathbf{k}) f_2^*(\mathbf{k})$$

Proof. We apply a Fourier transformation to function (2.1)

$$\int_{G_x} f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \int_{G_x} \left\{ \int_{G_y} f_1(\mathbf{x}-\mathbf{y}) f_2(\mathbf{y}) d\mathbf{y} \right\} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \tag{2.2}$$

Writing the right-hand side of Eq. (2.2) in the form of an integral over the region G in $2n$-dimensional space and replacing in this interval the variables $z_i = x_i - y_i$ and $y_i = y_i$ with a Jacobian equal to unity, we reduce it to the form

$$\iint_{G_{zy}} f_1(\mathbf{z}) f_2(\mathbf{y}) e^{-i(\mathbf{z}+\mathbf{y})\cdot\mathbf{k}} d\mathbf{z} d\mathbf{y} = \int_{G_z} f_1(\mathbf{z}) e^{-i\mathbf{k}\cdot\mathbf{z}} d\mathbf{z} \cdot \int_{G_y} f_2(\mathbf{y}) e^{-i\mathbf{k}\cdot\mathbf{y}} d\mathbf{y} \tag{2.3}$$

where G_{zy} is the region of the values of the new variables, and $G_z = G_{xy}$.

The theorem follows from relations (2.2) and (2.4).

3. THE QUADRATURE OF THE SOLUTION OF THE SECOND INITIAL-BOUNDARY-VALUE PROBLEM

We will consider the second initial-boundary-value problem of the dynamic theory of elasticity [4]

$$\begin{gathered} \sigma_{lm,m}(\mathbf{x},t) + F_l(\mathbf{x},t) = \rho \ddot{u}_l(\mathbf{x},t); \quad \varepsilon_{lj}(\mathbf{x},t) = (u_{l,j}(\mathbf{x},t) + u_{j,l}(\mathbf{x},t))/2 \\ \sigma_{lm}(\mathbf{x},t) = \Gamma_{lmpq} \varepsilon_{pq}(\mathbf{x},t); \quad \sigma_{lm}(\mathbf{x}_S,t) n_m(\mathbf{x}_S) = P_l(\mathbf{x}_S,t) \\ u_i(\mathbf{x},t=0) = u_{i0}(\mathbf{x}); \quad \dot{u}_i(\mathbf{x},t=0) = u_{i1}(\mathbf{x}) \end{gathered} \tag{3.1}$$

where $\sigma_{lm}(\mathbf{x},t)$, $\varepsilon_{pq}(\mathbf{x},t)$ and Γ_{lmpq} are the components of the stress and strain tensors and the constants of elasticity, $F_l(\mathbf{x},t)$ and $u_l(\mathbf{x},t)$ are the components of the mass force and the displacement vector, $u_{i0}(\mathbf{x})$ and $u_{i1}(\mathbf{x})$ are the initial distribution of the displacements and their velocities in the body being deformed, S is the surface bounding the body and $\mathbf{x}$ and t are the coordinates of the points of space and the time.

We apply a Laplace transformation with respect to time to Eqs (3.1)

$$\sigma^*_{lm,m}(\mathbf{x},p)+\rho p^2 u^*_l(\mathbf{x},p) = -F^*_l(\mathbf{x},p)+\rho(pu_{l0}(\mathbf{x})+u_{l1}(\mathbf{x}))$$

$$\sigma^*_{lm}(\mathbf{x},p)=\Gamma_{lmnq}\varepsilon^*_{nq}(\mathbf{x},p);\quad \varepsilon^*_{nq}(\mathbf{x},p)=(u^*_{n,q}(\mathbf{x},p)+u^*_{q,n}(\mathbf{x},p))/2 \tag{3.2}$$

$$\sigma^*_{lm}(\mathbf{x}_S,p)n_m(\mathbf{x}_S)=P^*_l(\mathbf{x}_S,p)$$

We will construct the quadrature of the solution of the initial-boundary-value problem.

For this, we will use the properties of the fundamental solution $R_{ln}(\mathbf{x}-\mathbf{y},p)$ of the equation of problem (3.2), according to which any of its particular solutions can be represented in the form [5]

$$u^*_l(\mathbf{x},p)=\int_W R_{ln}(\mathbf{x}-\mathbf{y},p)\Psi^*_n(\mathbf{y},p)d\mathbf{y} \tag{3.3}$$

We will place the body being deformed, which occupies a volume V, into a volume V_1 of larger size so that the surface S_1 bounding it does not have common points with the surface S. The volume comprising the difference between the volumes V and V_1 will be denoted by V_2: $V_2 = V_1 - V$; the volume V_2 is bounded by the surface S_2, where $S_2 = S_1 + S$.

Suppose the function $\Psi^*_n(\mathbf{y},p)$ in the volume V is identical with the function $\Phi^*_n(\mathbf{y},p)$ – the right-hand side of dynamic equation (3.2). Then, from relation (3.3) we obtain

$$u^*_l(\mathbf{x},p)=\int_V R_{ln}(\mathbf{x}-\mathbf{y},p)\Phi^*_n(\mathbf{y},p)d\mathbf{y}+\int_{V_2} R_{ln}(\mathbf{x}-\mathbf{y},p)\Psi^{1*}_n(\mathbf{y},p)d\mathbf{y} \tag{3.4}$$

In relation (3.4), $\Psi^{1*}_n(\mathbf{y},p)$ is the mass force specified in the volume V_2. We will define it in such a way that on the surface S the boundary conditions of problem (3.2) are satisfied. For this, we will construct the stresses

$$\sigma^*_{ij}(\mathbf{x},p)=\frac{1}{2}\Gamma_{ij\alpha\beta}\int_V\left[\frac{\partial}{\partial x_\alpha}R_{\beta n}(\mathbf{x}-\mathbf{y},p)+\frac{\partial}{\partial x_\beta}R_{\alpha n}(\mathbf{x}-\mathbf{y},p)\right]+\Phi^*_n(\mathbf{y},p)d\mathbf{y}+$$

$$+\frac{1}{2}\Gamma_{ij\alpha\beta}\int_{V_2}\left[\frac{\partial}{\partial x_\alpha}R_{\beta n}(\mathbf{x}-\mathbf{y},p)+\frac{\partial}{\partial x_\beta}R_{\alpha n}(\mathbf{x}-\mathbf{y},p)\right]\Psi^{1*}_n(\mathbf{y},p)d\mathbf{y} \tag{3.5}$$

We multiply Eq. (3.5) by $n_j(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$ and integrate over the surface S. We obtain

$$\int_S \sigma^*_{ij}(\mathbf{x}_S,p)n_j(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}dS=\Gamma^1_i(\mathbf{k},p)+\frac{1}{2}\Gamma_{ij\alpha\beta}\int_S n_j(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}\times$$

$$\times\left\{\int_{V_2}\left[\frac{\partial}{\partial x_\alpha}R_{\beta n}(\mathbf{x}-\mathbf{y},p)+\frac{\partial}{\partial x_\beta}R_{\alpha n}(\mathbf{x}-\mathbf{y},p)\right]\Psi^{1*}_n(\mathbf{y},p)d\mathbf{y}\right\}dS \tag{3.6}$$

Using the boundary conditions of problem (3.2), Gauss's theorem and the theorem on convolution over a finite region, with the notation

$$M_l(\mathbf{k},p)=P^{**}_l(\mathbf{k},p)-\Gamma^1_l(\mathbf{k},p)$$

$$N_{ln}(\mathbf{k},p)=-\frac{1}{2}\Gamma_{lj\alpha\beta}(ik_jR^*_{\beta n,\alpha}(\mathbf{k},p)+ik_jR^*_{\alpha n,\beta}(\mathbf{k},p)-R^*_{\beta n,\alpha j}(\mathbf{k},p)-R_{\alpha n,\beta j}(\mathbf{k},p))$$

$$P^{**}_l(\mathbf{k},p)=\int_S P^*_l(\mathbf{x}_S,p)e^{-i\mathbf{k}\cdot\mathbf{x}_S}dS$$

$$R^*_{\beta n,\alpha}(\mathbf{k},p)=\int_{V_z}\left[\frac{\partial}{\partial z_\alpha}R_{\beta n}(\mathbf{z},p)\right]e^{-i\mathbf{k}\cdot\mathbf{z}}d\mathbf{z},\quad R^*_{\alpha n,\beta j}(\mathbf{k},p)=\int_{V_z}\left[\frac{\partial^2}{\partial z_j\partial z_\beta}R_{\alpha n}(\mathbf{z},p)\right]e^{-i\mathbf{k}\cdot\mathbf{z}}d\mathbf{z}$$

$$\Gamma^1_i(\mathbf{k},p)=\frac{1}{2}\Gamma_{ij\alpha\beta}\int_S n_j(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}\left\{\int_V\left[\frac{\partial}{\partial x_\alpha}R_{\beta n}(\mathbf{x}-\mathbf{y},p)+\frac{\partial}{\partial x_\beta}R_{\alpha n}(\mathbf{x}-\mathbf{y},p)\right]\Phi^*_n(\mathbf{y},p)d\mathbf{y}\right\}dS$$

we reduce system of equations (3.6) to the form

$$M_l(\mathbf{k},p) = N_{ln}(\mathbf{k},p)\Psi_n^{1**}(\mathbf{k},p) \tag{3.7}$$

System of linear algebraic equations (3.7) is an integral form of boundary-value problem (3.2). Since boundary-value problem (3.2) possesses a unique solution [4], the matrix of system of equations (3.7) $N_{ln}(\mathbf{k}, p)$ has the inverse matrix $N_{nl}^{-1}(\mathbf{k}, p)$, which can be constructed by the well-known procedure. Therefore

$$\Psi_n^{1**}(\mathbf{k},p) = N_{nl}^{-1}(\mathbf{k},p)M_l(\mathbf{k},p) \tag{3.8}$$

Using Eqs (3.8) and (3.4), we construct the solution of problem (3.1)

$$u_l(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-\infty}^{\alpha+i\infty} e^{pt}\left\{\int_V R_{ln}(\mathbf{x}-\mathbf{y},p)\Phi_n^*(\mathbf{y},p)dy + \right.$$

$$\left. + \int_{V_2} R_{ln}(\mathbf{x}-\mathbf{y},p)\left[\frac{1}{(2\pi)^3}\int_{R^3} e^{i\mathbf{k}\cdot\mathbf{y}}N_{nq}^{-1}(\mathbf{k},p)M_q(\mathbf{k},p)dk_1dk_2dk_3\right]d\mathbf{y}\right\}dp \tag{3.9}$$

We will prove that quadrature (3.9) is the solution of initial-boundary-value problem (3.1).

Theorem 3. Quadrature (3.9) satisfies the equations of motion of initial-boundary-value problem (3.1).

Proof. For the proof, we will write Eq. (3.9) in the form

$$u_1(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt}u_1^*(\mathbf{x},p)dp \tag{3.10}$$

or

$$u_1(\mathbf{x},p) = \int_{V_1}\int_0^t R_{1j}^1(\mathbf{x}-\mathbf{y},t-\tau)\Phi_j^1(y,\tau)dyd\tau \tag{3.11}$$

$$R_{1j}^1(\mathbf{x}-\mathbf{y},t-\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{p(t-\tau)}R_{1j}(\mathbf{x}-\mathbf{y},p)dp, \quad \Phi_j^1(\mathbf{y},\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{p\tau}\Phi_j^{1*}(\mathbf{y},p)dp$$

Then, for the second derivative $\ddot{u}_1(\mathbf{x}, t)$, by the properties of the Laplace transformation, we obtain

$$\ddot{u}_1(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt}[u_1^*(\mathbf{x},p) - pu_{10}(\mathbf{x}) - u_{11}(\mathbf{x})]dp \tag{3.12}$$

Substituting expressions (3.11) and (3.12) into the equations of motion of initial-boundary-value problem (3.1), we obtain

$$\int_{V_1}\left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt}[(L)_{ml}R_{lj}(\mathbf{x}-\mathbf{y},p) + \rho p^2 R_{lj}(\mathbf{x}-\mathbf{y},p)]\Phi_j^{1*}(\mathbf{y},p)dp\right\}d\mathbf{y} +$$

$$+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt}\Phi_m^*(\mathbf{x},p)dp = 0 \tag{3.13}$$

Taking account of the fact that in relation (3.13) the expression in the square brackets is the kernel of the identity integral transformation, we obtain a true equality.

Theorem 4. Quadrature (3.9) satisfies the initial conditions of problem (3.1).

Proof. The proof is identical with the proof of Theorem 2 in [1].

Theorem 5. Quadrature (3.9) satisfies the boundary conditions of initial boundary-value problem (3.1).

Proof. The proof follows from the properties of the Laplace transformation and from the construction of the function $\Psi_n^{1*}(\mathbf{y}, p)$, given in the present paper.

4. QUADRATURE OF THE SOLUTION OF THE FIRST INITIAL-BOUNDARY-VALUE PROBLEM

We will consider the first initial-boundary-value problem of dynamic elasticity theory. It differs from problem (3.1) by the replacement of the first boundary condition with the condition

$$u_i(\mathbf{x}_S, t) = u_{i0}(\mathbf{x}_S, t)$$

This boundary-value problem differs from the problem examined earlier [1] only in the fact that here the elastic body is considered to be anisotropic. Therefore, to solve the problem, it is sufficient to use, instead of the Kupradze matrix, Green's tensor of the anisotropic problem. Then the quadrature of the solution of the problem takes the form

$$u_l(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{pt} \left\{ \int_V R_{lj}(\mathbf{x} - \mathbf{y}, p)[F_j^*(\mathbf{y}, p) - p u_{j0}(\mathbf{y}) - u_{j1}(\mathbf{y})] d\mathbf{y} + \right.$$

$$\left. + \int_{V_1} R_{lj}(\mathbf{x} - \mathbf{y}, p) \left[\frac{1}{(2\pi)^3} \int_{R^3} e^{i\mathbf{k}\cdot\mathbf{x}} (Res_{jm}(\mathbf{k}, p) u_{m0}^{**}(\mathbf{k}, p) - Res_{jm}(\mathbf{k}, p) \Phi_m^{**}(\mathbf{k}, p)) d\mathbf{k} \right] \right\} d\mathbf{y} dp$$

Repeating the arguments given earlier [1], we can prove that this solution satisfies the system of initial differential equations, and also the boundary and initial conditions of the problem.

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